Regressive and divergent functions on ordered and well-ordered sets

By ATTILA MÁTÉ

0. Introduction

This paper may be divided into three loosely connected, independently readable parts, though each part relies on a few concepts introduced at the beginning of the previous one(s). In the first one we shall extend two well-known theorems of W. NEUMER and G. FODOR describing stationary subsets of well-ordered sets, in that we shall consider subsets of an ordered set that is not necessarily well-ordered. The second part gives a simple and coherent proof of a theorem of R. M. SOLOVAY asserting that every stationary subset of an uncountable regular cardinal[1]) $\varkappa$ can be split into $\varkappa$ mutually disjoint stationary sets. Finally, the third part considers a property closely connected with stationarity of subsets of a singular cardinal.

1. Extension of the theorems of Neumer and Fodor

Consider an ordered set S. S will be called *Dedekind-complete* if each of its nonempty subsets that is bounded from above has a supremum in S. This is clearly equivalent to saying that every nonempty subset bounded from below of S has an infimum in S. It is well known that to every ordered set S there corresponds a set $S' \supseteqq S$ that is Dedekind-complete. There is such an S' minimal with respect to inclusion; this set is uniquely determined up to isomorphism. It is called the *Dedekind completion* of S and is denoted by $\mathbf{Dc}(S)$. A convenient way to describe the Dedekind completion of a set can be given in terms of the familiar Dedekind cuts. Well-known examples for Dedekind-complete sets are the set of all real numbers and any well-ordered set.

Assume V is an ordered set and denote the ordering of V simply by $<$. Call a subset X of $\mathbf{Dc}(V)$ a *V-band* if X is not bounded in $\mathbf{Dc}(V)$ from above and X is

[1]) Cardinals are identified with their initial ordinals, and ordinals with the sets of all their predecessors.

closed upward in $\mathbf{Dc}(V)$, i.e. for any subset X' bounded from above of X, we have $\sup X' \in X$, where the supremum is taken in $\mathbf{Dc}(V)$. Call a subset X of V *V-stationary* if X meets every V-band. Call a function f *V-regressive* if f maps a subset of V into V and, with some v_0 depending on f, we have $f(x) < x$ whenever x belongs to the domain of f and $x \geqq v_0$. Call f *V-divergent* if f maps a subset of V into V and for no $v \in V$ is the set

$$\{x \in V : f(x) < v\}$$

cofinal to V. Then our generalization of Neumer's theorem runs as follows (cf. [5, Sätze 2 and 4 on p. 257], [2, Theorem on p. 204], and [1, Sätze 1 and 2 on p. 46]):

Theorem 1. 1. *Assume V is not cofinal to any ordinal $\leqq \aleph_0$.*[2]) *Then $X \subseteq V$ is not V-stationary if and only if there is a V-regressive and V-divergent function f with domain X.*

The validity of the theorem can be extended to the pathological case in which V is cofinal to $\aleph_0$ if we redefine (as it is often useful to do) the notion of stationarity in such a way that, for V cofinal to $\aleph_0$, we call every subset of V V-nonstationary (i.e. not V-stationary).

Proof. We may assume that X is cofinal to V; otherwise the assertion would be ovious. First we establish the

"if" part. Assume that f is a V-regressive and V-divergent function on X. Put

$$g(u) = \inf \{f(x) : x \in X \,\&\, u \leqq x\}$$

for any $u \in \overline{V}$, where $\overline{V}$ Denotes the Dedekind completion $\mathbf{Dc}(V)$ of V and the infimum is taken in $\overline{V}$. Since f is V-divergent, for large $u \in \overline{V}$[3]) the set on the right-hand side in the last centred line is bounded from below; so the infimum exists. For small (i.e. not large) $u \in \overline{V}$ we may achieve that this infimum exist by changing the values of f assumed for small $x \in X$ (this does not affect the relevant properties of f).

The function g is defined for all elements of V and is obviously monotonic. For any element x of X we have

$$g(x) \leqq f(x).$$

So the set

$$B = \{u \in \overline{V} : g(u) \geqq u\}$$

intersects X only in a set not cofinal to $\overline{V}$. In view of the monotonity of g, B is obviously closed upward. Now, we have two alternatives: either

a) B is cofinal to $\overline{V}$, or *b)* B is not cofinal to $\overline{V}$.

---

[2]) I.e. V neither contains a last element nor is cofinal to $\aleph_0$.

[3]) I.e. for any $u > u_0$ with a fixed $u_0 \in V$, possibly depending on f.

In case *a)*, B is a $\bar{V}$-band (or, what is the same, a V-band) that is disjoint from X (in essence, i.e. if we disregard a set not cofinal to $\bar{V}$). So X is not V-stationary, which was to prove.

In case *b)*, g is a $\bar{V}$-regressive and $\bar{V}$-divergent function defined on the whole of $\bar{V}$. Assume $v_0 \in \bar{V}$ is such that for any $u \geqq v_0$ we have $g(u) < u$. If, for an integer k, v_k has already been defined, then choose v_{k+1} such that $v_k < g(v_{k+1})$. Put

$$v_\omega = \sup \{v_k : k < \aleph_0\} (= \sup \{g(v_k) : k < \aleph_0\}).$$

(The definition of v_ω is sound since $\bar{V}$ is Dedekind-complete and is not cofinal to $\aleph_0$.) Then the monotonity of g implies $g(v_\omega) \geqq v_\omega$, which is a contradiction. This completes the proof of the "if" part of the theorem. Now we are proving the

"only if" part. For any elements u and v of $\bar{V}$ choose $w = h(u, v)$ such that

$$w \in V \quad \text{and} \quad u \leqq w < v \quad \text{whenever} \quad u < v$$

(otherwise put e.g. $h(u, v) = v$). This is possible: Indeed assume $u < v$. If $u \in V$ then we may simply take $w = u$; and if $u \notin V$ then, u being an element of the Dedekind completion $\bar{V}$ of V, it can be represented as the infimum (in $\bar{V}$) of a suitable subset W of V. A small enough element of this set can be chosen as w.

Now, assuming that $B \subseteqq \bar{V}$ is a V-band disjoint from the V-nonstationary set $X \subseteqq V$, put

$$g(x) = \sup \{b \in B : b < x\},$$

for any $x \in X$. Then g is obviously $\bar{V}$-regressive and $\bar{V}$-divergent on X, the only trouble being that its values are not necessarily in V. So, putting

$$f(x) = h(g(x), x)$$

for every $x \in X$, the function f is V-regressive and V-divergent, completing the proof.

The next theorem is a generalization of an important theorem of FODOR (see [3, Satz 2 on p. 141]):

Theorem 1.2. *Assume that V is not cofinal to any ordinal $\leqq \aleph_0$, X is a V-stationary subset of V, and f is a V-regressive function on X. Then, for some $v \in V$ the set*

$$f^{-1}(\leqq v) \overset{\text{def}}{=} \{x \in X : f(x) \leqq v\}$$

is V-stationary.

Proof. A well-known theorem of Hausdorff says that every ordered set is cofinal to one of its well-ordered subsets; this is actually an easy consequence of Zermelo's Well Ordering Theorem. So, take an increasing sequence $\{v_\xi : \xi < \alpha\}$ that is cofinal to V; here choose α the least possible ordinal, which then may be called the *cofinality number* of V.

Assuming that for no v_ξ $(\xi<\alpha)$ is the set $X_\xi=f^{-1}(\leqq v_\xi)$ V-stationary, the preceding theorem ensures the existence of a V-regressive and V-divergent function f_ξ with domain X_ξ. Here, without any restriction of generality, we may assume that e.g.

$$f_\xi(x)<x \quad \text{whenever} \quad x>v_0 \quad \text{and} \quad \xi<\alpha. \tag{1. 1}$$

Now, for any $\xi<\alpha$ put

$$Y_\xi = X_\xi - \bigcup_{\eta<\xi} X_\eta \left(=\{x\in X\colon (\forall\eta<\xi)(v_\eta<f(x)\leqq v_\xi)\}\right),$$

and, for $x\in Y_\xi$ write

$$g(x)=\max\left(f(x), f_\xi(x)\right).$$

Since $\bigcup_{\xi<\alpha} Y_\xi=X$, the domain of g is X. g is obviously V-regressive (cf. (1. 1)). We are going to show that it is also V-divergent. Indeed, for any $\lambda<\alpha$ we have:

$$\{x\in X\colon g(x)\leqq v_\lambda\} = \{x\in X\colon f(x)\leqq v_\lambda\}\cap \bigcup_{\xi<\alpha}\{x\in Y_\xi\colon f_\xi(x)\leqq v_\lambda\}=$$
$$=\bigcup_{\xi\leqq\lambda}\{x\in Y_\xi\colon f_\xi(x)\leqq v_\lambda\}.$$

By the divergence of f_ξ, none of the sets on the rightmost side here is cofinal to V. Since their number is less than α, which was chosen to be the cofinality number of V, neither is so their union. Since the sequence $\{v_\xi\colon \xi<\alpha\}$ is cofinal V, this shows that g is V-divergent. So the preceding theorem implies that X is V-nonstationary, in contradiction with our assumptions. The theorem is proved.

The above extended notion of stationary sets is closely connected with the classical one. To locate a point of contact we derive the following.

Theorem 1. 3. *Assume the cofinality of V is $>\aleph_0$. Assume, further, that X is a subset of V and $B\subseteqq \overline{V}(=\mathbf{Dc}(V))$ is a V-band. Then X is V-stationary if and only if $X\cap B$ is B-stationary.*

Proof. We only have to prove that the set $Y=\overline{V}-X$ includes a V-band if and only if $Y\cap B$ includes a B-band. Now, if $C\subseteqq Y\cap B$ is a B-band then C is obviously also a V-band included in Y. To see this we only have to observe that C is closed upward in $\overline{V}$. Conversely, assume that $D\subseteqq Y$ is a V-band. Then $D\cap B\subseteqq$ $\subseteqq Y\cap B$ is clearly closed upward in B. So it is a B-band, since it is also unbounded; this latter can be seen directly, but also follows immediately from the fact that the union of two $\overline{V}$-nonstationary sets (namely $\overline{V}-B$ and $\overline{V}-D$) is also V-nonstationary, which can be seen e.g. by invoking Theorem 1. 1. The proof is complete.

Now the theorem of Hausdorff mentioned at the beginning of the proof of Theorem 1. 2 entails the existence of a well-ordered V-band; so our last result establishes a simple connection between stationary sets in the extended sense and those in well-ordered sets.

2. Solovay's decomposition theorem

One of the nicest results in the theory of stationary sets is the following one conjectured by FODOR and proved by SOLOVAY (see [8, p. 418]; cf. also [6] and [7]):

Theorem 2. 1. *Assume $\varkappa$ is a regular cardinal $>\aleph_0$. Then every $\varkappa$-stationary set can be split into $\varkappa$ mutually disjoint $\varkappa$-stationary sets.*

Several weaker results had previously been obtained by G. FODOR, A. HAJNAL, A. LÉVY, and R. M. SOLOVAY. Here the definition of *$\varkappa$-stationary* sets can be obtained from the definition given in the preceding section if we take $V=\varkappa$ as ordered by the natural ordering of ordinals. Every well-ordered set being Dedekind-complete, the situation is simplified by the fact that in this case we have $\mathbf{Dc}(V)=V$. Therefore we do not have to distinguish between V- and $\mathbf{Dc}(V)$-regressive functions; so it will not lead to confusion if we simply speak of regressive functions.

To prove this theorem we need a few concepts. First of all, call a set $\mathscr{I}$ of subsets of $\varkappa$ a *$\varkappa$-complete ideal* carried by $\varkappa$ if any subset of an element of $\mathscr{I}$ as well as any union of a number less than $\varkappa$ of elements of $\mathscr{I}$ belongs to $\mathscr{I}$. $\mathscr{I}$ is said to be *proper* if $\varkappa\notin\mathscr{I}$. Call $\mathscr{I}$ *$\varkappa$-saturated* if there are no $\varkappa$ mutually disjoint subsets of $\varkappa$ none of which belongs to $\mathscr{I}$. Call $\mathscr{I}$ a *normal* ideal carried by $\varkappa$ if it is a $\varkappa$-complete proper ideal carried by $\varkappa$ that contains each one-element subset of $\varkappa$ and is such that every regressive function defined on a subset of $\varkappa$ not in $\mathscr{I}$ is constant on a set outside $\mathscr{I}$. Finally, for a class A of ordinals denote by $\mathbf{nst}(A)$ the class of all ordinals *nonstationary* with respect to A; i.e. $\alpha\in\mathbf{nst}(A)$ if and only if either the cofinality of α is $\aleph_0$ or its cofinality is $>\aleph_0$ and $A\cap\alpha$ is not α-stationary.[4])

Solovay's proof of Theorem 2. 1 is based on Lemmas 2. 2—2. 4 below. In their formulations, $\varkappa$ denotes a fixed regular cardinal $>\aleph_0$. The proofs of the first two of these given here are due to SOLOVAY; our innovation is only in the proof of the third one; our proof is in effect based on a reduced-product argument.

Lemma 2. 2. *If $A\subseteq K$ is $\varkappa$-stationary then so is $A\cap\mathbf{nst}(A)$.*

Proof.[5]) Assuming that the set $A\cap\mathbf{nst}(A)$ is $\varkappa$-nonstationary, there exists $\varkappa$-band B that is disjoint from it. The set $B'\subseteq B$ of all limit points of B is also $\varkappa$-band, so it intersects A. The first point α in $B'\cap A$ belongs to $\mathbf{nst}(A)$; indeed, if α is cofinal to $\aleph_0$ then this is by definition so; if not, then $B'\cap\alpha$ is an α-band disjoint from $A\cap\alpha$, showing that $A\cap\alpha$ is α-nonstationary. Since $\alpha\in B$, this contradicts the assumption that B and $A\cap\mathbf{nst}(A)$ are disjoint, completing the proof.

---

[4]) Clearly, $\mathbf{nst}(A)$ is always a real class.

[5]) The proofs of this and the next lemma are reproduced here with R. M. SOLOVAY's permission.

Lemma 2. 3. *If $\mathcal{I}$ is a $\varkappa$-saturated normal ideal carried by $\varkappa$, then every regressive function f defined on a subset $X \notin \mathcal{I}$ of $\varkappa$ is bounded on the whole of its domain with the possible exception of a set $N \in \mathcal{I}$. (Shortly: f is essentially bounded.)*

Proof. Let Z be the set of all ordinals $\zeta < \varkappa$ such that

$$X_\zeta = \{\alpha \in X \colon f(\alpha) = \zeta\} \notin \mathcal{I}.$$

Then, by the $\varkappa$-saturatedness of $\mathcal{I}$, the cardinality of Z is less than $\varkappa$. On the other hand, by the normality of $\mathcal{I}$ we have $X - X' \in \mathcal{I}$ with $X' = \bigcup_{\zeta \in Z} X_\zeta$. For $\alpha \in X'$ we have $f(\alpha) \leqq \bigcup Z < \varkappa$. This completes the proof.

Lemma 2. 4. *Assume $\mathcal{I}$ is a $\varkappa$-saturated normal ideal carried by $\varkappa$, and $A \notin \mathcal{I}$. Then we have $\mathbf{nst}(A) \cap \varkappa \in \mathcal{I}$.*

Proof. By Neumer's theorem (cf. Theorem 1. 1 and the remark thereafter), for any α in $S \overset{\text{def}}{=} \mathbf{nst}(A) \cap \varkappa$ there exists an α-divergent regressive function f_α on $A \cap \alpha$. We may assume that $f_\alpha(\gamma) < \gamma$ holds for any $\gamma \neq 0$ in $A \cap \alpha$. Assuming that S does not belong to $\mathcal{I}$, for any $\gamma \in A$ write

$$f(\gamma) = \min \{\beta : \{\alpha \in S : \alpha > \gamma \,\&\, f_\alpha(\gamma) = \beta\} \notin \mathcal{I}\}.$$

Since $\mathcal{I}$ is a $\varkappa$-complete ideal, this definition is sound, and, obviously, f is regressive. So, by the preceding lemma, we have

$$f(\gamma) < \xi \quad \text{whenever} \quad \gamma \in A - M, \tag{2. 1}$$

with a suitable ordinal $\xi < \varkappa$ and a set $M \in \mathcal{I}$.

Now, for any $\alpha \in S$ with $\alpha > \xi$ put

$$g(\alpha) = \bigcup \{\gamma \in A \cap \alpha \colon \; f_\alpha(\gamma) < \xi\}.$$

The α-divergence of f_α implies that $g(\alpha) < \alpha$ for any α in the domain of g. So, again by the preceding lemma, we have

$$g(\alpha) < \lambda \quad \text{whenever} \quad \alpha \in S - N,$$

with a suitable $\lambda < \varkappa$ and $N \in \mathcal{I}$. Taking the definition of g also into account, we see from here that $f_\alpha(\gamma) \geqq \xi$ whenever $\alpha \in S - N$, $\gamma \in A$, and $\lambda \leqq \gamma < \alpha$. Therefore, the definition of f implies that $f(\gamma) \geqq \xi$ for all $\gamma \in A$ with $\gamma \geqq \lambda$. This contradicts (2. 1), proving the lemma.

Theorem 2. 1 is a simple consequence of Lemmas 2. 2 and 2. 4. Indeed, in the latter lemma take $\mathcal{I}$ as the set of all subsets X of $\varkappa$ such that $X \cap A$ is $\varkappa$-nonstationary. Then $\mathcal{I}$ is a normal ideal in view of Fodor's theorem (cf. Theorem 1. 2); and, if we assume that Theorem 2. 1 fails for A, then $\mathcal{I}$ is also $\varkappa$-saturated. Now, the assertions of the two lemmas contradict each other, proving Theorem 2. 1.

With the aid of Theorem 1. 3, from Theorem 2. 1 we may easily derive the following

Corollary 2. 5. *Assume V is an ordered set, and $\varkappa > \aleph_0$ is the cofinality number of its order type. Then any V-stationary subset of V can be split into $\varkappa$ mutually disjoint V-stationary sets.*

Another result, in a sense running counter to Theorem 1. 3, may also be derived:

Corollary 2. 6. *Assume $\varkappa$ is an ordinal,* $\mathrm{cf}(\varkappa) > \aleph_0$.[6]) *Then there exist sets S and T with $S \subseteqq T \subseteqq \varkappa$ such that S is T-stationary, T is $\varkappa$-stationary, and yet S is not $\varkappa$-stationary.*

Proof. Choose $T \subseteqq \varkappa$ such that T and $\varkappa - T$ are $\varkappa$-stationary; this is possible e.g. by the preceding corollary.[7]) For any $\alpha \in T$ put

$$h(\alpha) = \min\{\beta : \forall \xi (\beta \leqq \xi < \alpha \rightarrow \xi \notin T)\}.$$

Then we obviously have $h(\alpha) \leqq \alpha$, and h is a $\varkappa$-divergent function. Set

$$S = \{\alpha \in T : h(\alpha) < \alpha\}.$$

In view of the divergence of h, S is $\varkappa$-nonstationary by Neumer's theorem (cf. Theorem 1. 1).

We are going to show that S is also T-stationary. Indeed, assume, on the contrary, that there exists a regressive $\varkappa$-divergent function f mapping S into T. Then the function

$$g(\gamma) = f(\eta), \quad \text{where} \quad \eta = \min\{\vartheta \in S : \vartheta > \gamma\},$$

defined for every $\gamma \in \varkappa - T$, is also regressive and $\varkappa$-divergent, in contradiction with the $\varkappa$-stationarity of this set. The proof is complete.

3. Solid sets

Assume $\varkappa$ is a singular cardinal of cofinality $> \aleph_0$. Call a set $X \subseteqq \varkappa$ $\varkappa$-solid if there is no $\varkappa$-band of cardinality $\varkappa$ disjoint from X. It is easy to see that a $\varkappa$-solid set is not necessarily $\varkappa$-stationary; but how can such sets be characterized? A kind of answer to this question is given by

---

[6]) $\mathrm{cf}(\varkappa)$ denotes the cofinality of $\varkappa$, i.e. the least cardinal cofinal to $\varkappa$.

[7]) The existence of such a T can also be shown by much simpler arguments.

Theorem 3. 1. *Assuming that $\varkappa$ is a singular cardinal with* $\mathrm{cf}(\varkappa)>\aleph_0$, *a set* $X\subseteqq\varkappa$ *is $\varkappa$-solid if and only if either X is $\varkappa$-stationary or*

$$\mathbf{U}\{\mathrm{cf}(\xi):\xi\in\mathbf{nst}(X)\}<\varkappa. \tag{3. 1}$$

Proof. "*If*". Assume that X is not $\varkappa$-solid, and choose a $\varkappa$-band B of cardinality $\varkappa$ that is disjoint from X. Then, for any regular cardinal $\alpha<\varkappa$, the αth element of B, which obviously belongs to $\mathbf{nst}(X)$ (all limit points of B do), is of cofinality α; so (3. 1) does not hold. This completes the proof of the "if" part.

"*Only if*". Assuming that is X not $\varkappa$-stationary, there exists a $\varkappa$-band

$$B=\{\beta_\gamma\}_{\gamma<\mathrm{cf}(\varkappa)}$$

included in $\varkappa-X$. If, furthermore, we assume that (3. 1) does not hold, we see the existence of a sequence

$$\{\xi_\gamma\}_{\gamma<\mathrm{cf}(\varkappa)}\subseteqq\mathbf{nst}(X)$$

of ordinals ξ_γ such that

$$\eta_\gamma=\mathrm{cf}(\xi_\gamma)\ \text{tends to}\ \varkappa\ \ (\gamma<\mathrm{cf}(\varkappa)).$$

We may assume that the sequence $\{\eta_\gamma\}_{\gamma<\mathrm{cf}(\varkappa)}$ is increasing and $\eta_0>\aleph_0$.

So, for each $\gamma<\mathrm{cf}(\varkappa)$ there exists a ξ_γ-band

$$S_\gamma=\{\sigma_{\gamma,\delta}\}_{\delta<\eta_\gamma}$$

included in $\xi_\gamma-X$. Put

$$S'_\gamma=\{\sigma_{\gamma+1,\delta}\}_{\delta\leqq\eta_\gamma}.$$

Then S'_γ is obviously a closed set.

Now, for any ordinal $\gamma<\mathrm{cf}(\varkappa)$ denote by λ_γ the least ordinal $\lambda<\mathrm{cf}(\varkappa)$ such that $\sigma_{\lambda+1,\eta_\lambda}>\beta_\gamma$, and set

$$S=B\cup\mathbf{U}\{S'_{\lambda_\gamma}-\beta_\gamma:\gamma<\mathrm{cf}(\varkappa)\}.$$

Then S is obviously a $\varkappa$-band of cardinality $\varkappa$ that is included in $\varkappa-X$. This shows that X is not $\varkappa$-solid, completing the proof.

References

[1] H. Bachmann, *Transfinite Zahlen*, 2nd ed. (Berlin—Heidelberg—New York, 1967.)

[2] G. Fodor, Generalization of a theorem of Alexandroff and Urysohn, *Acta Sci. Math.*, **16** (1955), 204—206.

[3] G. Fodor, Eine Bemerkung zur Theorie der regressiven Funktionen, *Acta Sci. Math.*, **17** (1956), 139—142.

[4] G. Fodor and A. Máté, Some results concerning regressive functions, *Acta Sci. Math.*, **30** (1969), 247—254.

[5] W. NEUMER, Verallgemeinung eines Satzes von Alexandroff und Urysohn, *Math. Z.*, **54** (1951), 254—261.

[6] R. M. SOLOVAY, Real-valued measurable cardinals (Xerographed notes written by A. R. D. Mathias and edited by D. Scott), *University of California*, Berkeley, 1967.

[7] R. M. SOLOVAY, Solution of a problem of Fodor and Hajnal, *ibidem*.

[8] R. M. SOLOVAY, Real-valued measurable cardinals, in: Axiomatic Set Theory, *Proceedings of Symposia in Pure Math.*, Vol. XIII, Part I, Amer. Math. Soc. (Providence, Rhode Island, 1971), 397—428.

BOLYAI INSTITUTE
SZEGED, HUNGARY

(Received April 15, 1971)